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Guaranteed state estimation by zonotopes

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***Abstract*— This paper presents a new approach to guaranteed state estimation for non linear discrete-time systems with a bounded description of noise and parameters. The main result is an algorithm to compute a set that contains the states consistent with the measured output and the given noise and parameters. This set is represented by a zonotope. The volume of the zonotope is minimized each sample instant solving a convex optimization problem. Interval arithmetic is used to calculate a guaranteed trajectory of the state process. Two examples have been provided for clarifying the algorithm.**

Keywords: Estimation, Observers for nonlinear systems, Optimization algorithms

I. INTRODUCTION

The problem of state estimation is important in the theory of identification. Several approaches exist in the literature. The stochastic approach (Kalman filter theory), the H_∞ filtering theory and the set-membership approach can be noted.

The Kalman filter gives an optimal estimate of the state of a given process based on output measurements. However, an accurate model of the process and covariance matrices of noise are needed. In this context, it is possible to design a filter minimizing the worst-case gain of the system [11], [14]. See [1], [2] for an extension of the classical Kalman filtering to interval linear systems.

The set-membership approach is based on the construction of a compact set that includes, with guarantee, the states of the system that are consistent with the measured output and the bounded noise. Instead of using Gaussian noise, as in the stochastic approach, a norm-bounded noise is considered. In this paper, this approach has been adopted.

In pioneers works about set-membership estimation [17], an ellipsoidal bounding of the state of the dynamic system is provided. The advantage of such a choice lies in the fact that the computational complexity of the corresponding estimation does not depend on the numbers of observations. The application of ellipsoidal sets to the state estimation problem has been studied by different authors. (See, for example, [10], [16], [4]) .

In order to obtain an increased estimation accuracy, the use of polyhedrons was proposed [9]. Taking into account the fact the complexity of this representation grows considerably with the volume of measured information, an alternative approach based on parallelotopes was presented [3]. Minimum-volume bounding parallelotopes are used to estimate the state of a discrete-linear dynamical system with polynomial complexity. In [7] a guaranteed recursive nonlinear estimator is based on an interval branch and bound algorithm. To improve the exponential complexity, consistency techniques are considered in [6]. Zonotopes are proposed in [15] to build a worst-case state estimator.

In this paper, a new method for guaranteed state estimation for the case of non-linear discrete process with bounded uncertain parameters and noise is presented. Each sample time, a guaranteed bound of the uncertain trajectory of the system is calculated using interval arithmetic. Then, a convex optimization problem is formulated to compute the intersection between the bound of the uncertain trajectory and the states of the system that are consistent with the measured output. Like in [15], the calculated set is represented by a zonotope. However, in [15] the measured output is used to estimate the state by means of a gain K, while here it is used to calculate the region of the state space that are consistent with the measured outputs.

The paper is organized as follows. In section 2 the problem is formulated. The guaranteed state estimation algorithm is presented in section 3. Section 4 provides a revision of the Kühn's method [8] to compute trajectories of non-linear systems. Sections 5 and 6 provide a bound of the consistent state. Section 7 reports two examples. The paper draws to a close with a section of conclusions.

II. PROBLEM FORMULATION

Consider an uncertain nonlinear discrete-time system of the form:

$$\begin{aligned} x_{k+1} &= f(x_k, w_k) \\ y_k &= g(x_k, v_k) \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state of the system and $y_k \in \mathbb{R}^p$ is the measured output vector at sample time k. The vector $w_k \in \mathbb{R}^{n_w}$ represents the time varying process parameters and process perturbation vector. On the other hand, $v_k \in \mathbb{R}^{p_v}$ is the measurement noise vector. It is assumed that the uncertainties and initial state are bounded by known compact sets: $w_k \in W$, $v_k \in V$ and $x_o \in \mathcal{X}_0$. Then, at sample time k, the objective is to find an outer approximation of the corresponding set of all possible states consistent with the measured outputs and the initial state set.

Definition 1 (Exact uncertain state set): Consider a system given by equation (1) and an initial compact state set $\mathcal{X}_0$. Consider also a sequence of measured outputs $(y_i)_1^k$. Then, at sample time k the uncertain state set $\mathcal{X}_k$ is defined as:

$$\mathcal{X}_k = \{\, x_k \,:\, \exists (w_j)_0^{k-1} \in W, (v_j)_1^k \in V, x_0 \in \mathcal{X}_0$$
$$\text{such that } (x_j = f(x_{j-1}, w_{j-1}), y_i = g(x_j, v_j))_1^k \}$$

So, $\mathcal{X}_k$ is the set of all states, consistent with the measured output, that can be reached by the evolution of the uncertain system at sample time k. The exact computation of these sets is a difficult task. In order to reduce the complexity of the computations, these sets are bounded by means of conservative sets. These approximate sets, denominated uncertain state sets, may be computed more easily.

III. Guaranteed state estimation algorithm

In this section we introduce some definitions that allows us to state a general procedure to obtain a bound of the exact uncertain state set.

Definition 2 (Range): The range of a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ over a set $X \subset \mathbb{R}^n$ is defined as $f(X) = \{\, f(x) \,:\, x \in X \,\}$.

Definition 3 (Inclusion function): Ψ is a inclusion function of $f(.)$ if $f(X) \subseteq \Psi(X)$, $\forall X$.

Definition 4 (Consistent state set): Given system (1) and a measured output y_k, the consistent state set is defined by $\mathcal{X}_{y_k} = \{\, x \in \mathbb{R}^n \,:\, y_k \in g(x, V) \,\}$.

Let us suppose that at sample time k a measured output y_k is obtained and that a bound of the exact uncertain state set at time $k-1$ is available (this bound will be denoted $\tilde{\mathcal{X}}_{k-1}$). Then, the following algorithm estimates a bound of the exact uncertain state set:

Algorithm 1

Step 1: Use an inclusion function to bound the uncertain trajectory of the nonlinear system: $\bar{\mathcal{X}}_k = \Psi(\tilde{\mathcal{X}}_{k-1}, W)$.
Step 2: Compute a bound of the consistent state set $\mathcal{X}_{y_k}^e \supseteq \mathcal{X}_{y_k}$.
Step 3: Compute a tight bound of the set intersection $\bar{\mathcal{X}}_k \cap \mathcal{X}_{y_k}^e \subseteq \tilde{\mathcal{X}}_k$.

End algorithm

Theorem 1: At sample time k, consider a system given by (1), a measured output y_k, a bound of the uncertain state set at time $k-1$ (denoted $\tilde{\mathcal{X}}_{k-1}$). Suppose that $\tilde{\mathcal{X}}_k$ is computed by Algorithm 1 then, $\tilde{\mathcal{X}}_k$ is an outer approximation of the exact uncertain state set $\mathcal{X}_k \subseteq \tilde{\mathcal{X}}_k$.

PROOF:

$$\mathcal{X}_k = f(\mathcal{X}_{k-1}, W) \cap \mathcal{X}_{y_k} \subseteq f(\tilde{\mathcal{X}}_{k-1}, W) \cap \mathcal{X}_{y_k}^e \subseteq \tilde{\mathcal{X}}_k$$

■

To obtain the uncertain trajectory of the non-linear discrete time system in Step 1, we propose to use Künh's method. This method represents the uncertain sets by zonotopes [8]. To compute the set intersection of Step 3, an optimization problem that minimizes the volume of $\tilde{\mathcal{X}}_k$ will be presented in section 5.

IV. Computing uncertain trajectories

A. Interval arithmetic

An interval number $X = [a, b]$ is the set $\{\, x \,:\, a \le x \le b \,\}$ of real numbers between and including the endpoints a and b. Interval arithmetic is an arithmetic defined on sets of intervals, rather than sets of real numbers. The interval arithmetic is based on operations applied to sets of intervals.

Let $\mathbb{I}$ be the set of real compact intervals $[a, b]$ with $a, b \in \mathbb{R}$. Operations in $\mathbb{I}$ satisfy the expression:

$$A \; op \; B = \{\, a \; op \; b \,:\, a \in A, b \in B \,\} \tag{2}$$

In this way, the four basic interval operations [13] are:

$$[a,b] + [c,d] = [a+c, b+d] \tag{3}$$
$$[a,b] - [c,d] = [a-d, b-c]$$
$$[a,b] * [c,d] = [\min(ac,ad,bc,bd), \max(ac,ad,bc,bd)]$$
$$[a,b]/[c,d] = [a,b] * [1/d, 1/c], \text{ if } 0 \notin [c,d]$$

An extension of the interval arithmetic to include 0 in division can be found in [5]. The interval extension of standard functions $\{sin, cos, tan, arctan, exp, ln, abs, sqr, sqrt\}$ is possible too.

Definition 5 (Unitary interval): The unitary interval is $\mathbf{B} = [-1, 1]$.

Definition 6 (Box): A box is an interval vector. An interval hull of a set $X \subseteq \mathbb{R}^n$, denoted by $\square X$, is a box that satisfies $X \subseteq \square X$. Given a box $\square X = ([a_1, b_1], \ldots, [a_n, b_n])^\top$, $mid(\square X)$ denotes its center and $diam(\square X) = (b_1 - a_1, \ldots, b_n - a_n)^\top$.

Definition 7 (Unitary box): A unitary box, denoted by $\mathbf{B}^m$, is a box compound by m unitary intervals.

Definition 8 (Natural interval extension): If $f : \mathbb{R}^n \to \mathbb{R}$ is a function computable as an expression, algorithm or computer program involving the four elementary arithmetic operations interspersed with evaluations of standard functions then, a natural interval extension of f, denoted $\square f$, is obtained replacing each occurrence of each variable by the corresponding interval variable, by executing all operations according to formulas (3) and by computing ranges of the standard functions. [7]

Theorem 2: A natural interval extension $\square f$ of a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ over a box $X \subseteq \mathbb{R}^n$ satisfies that $f(X) \subseteq \square f(X)$. This is the fundamental theorem of the interval arithmetic [12].

Theorem 3 (Mean Value Theorem): Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at every point in an open set containing the line segment L joining two vectors $x, y \in \mathbb{R}^n$. There is a vector $x_0 \in L$ such that: $f(x) - f(y) = \nabla f(x_0)(x - y)$. Suppose $X \in \mathbb{I}^n$ such $x, y \in X$ then applying the fundamental theorem of the interval arithmetic: $f(x) \in f(y) \oplus \nabla f(X)(x - y)$. This is the mean value extension.

Definition 9 (Mean value extension): Suppose a function $f : \mathbb{R}^n \to \mathbb{R}$ with continuous derivatives in $X \in \mathbb{I}^n$. Suppose also a real vector $c \in X$. Then, the mean value extension for f over X is defined by $f_{mve}(X) = f(c) \oplus \Box\nabla_x f(X)(X-c)$, where $\Box\nabla_x f(X)$ is an interval enclosure for the range of $\nabla f(X)$ over X.

B. Künh´s method

The Künh´s method is a procedure that allows us to bound the orbits of discrete dynamical systems [8]. The evolution of the system is approximated by a high order zonotope. A zonotope is the Minkowski sum of a set of parallelepipeds. In [8] sub-exponential overestimation is proven. These concepts are defined below.

Definition 10 (Minkowski sum): The Minkowski sum of two sets X and Y is defined by $X \oplus Y = \{\, x+y \,:\, x \in X, y \in Y \,\}$.

Definition 11 (Zonotope of order m): A zonotope Z of order m is the Minkowski sum of m parallelepipeds: $Z = P_1 \oplus P_2 \oplus \ldots \oplus P_m$

A parallelepiped is a linear image $P = M\mathbf{B}^q$ where M is a square matrix and $\mathbf{B}^q$ is the unitary box in $\mathbb{R}^q$. The order m is a measure for the geometrical complexity of the zonotopes.

Theorem 4 (Zonotope inclusion): Given a family of zonotopes represented by $Z = p \oplus M\mathbf{B}^m$ where $p \in \mathbb{R}^n$ is a real vector, $M \in \mathbb{I}^{n\times m}$ is an interval matrix and $\mathbf{B}^m \in \mathbb{I}^m$ is a unitary box. A zonotope inclusion, denoted by $\diamond Z$, is defined by $\diamond Z = p \oplus \begin{bmatrix} mid(M) & G \end{bmatrix} \begin{bmatrix} \mathbf{B}^m \\ \mathbf{B}^n \end{bmatrix} = p \oplus J\mathbf{B}^{m+n}$ where $G \in \mathbb{R}^{n\times n}$ is a diagonal matrix that satisfies $G_{ii} = \sum_{j=1}^{m} \frac{diam(M_{ij})}{2}$, $i = 1,\ldots,n$. Under these definitions it results that:

$$Z \subseteq \diamond Z$$

PROOF:

Let us suppose that $z \in Z$. Then, it is clear that there is $b \in \mathbf{B}^m$ such that: $z \in p \oplus Mb$. In fact, adding and substracting $mid(M)b$:

$$z \in (p + mid(M)b) \oplus (M - mid(M))b$$

It is not difficult to see that $(M - mid(M))b \subseteq G\mathbf{B}^n$. Thus,

$$z \in (p + mid(M)b) \oplus G\mathbf{B}^n \subseteq p \oplus mid(M)\mathbf{B}^m \oplus G\mathbf{B}^n = \diamond Z$$

■

Theorem 5: Given a function $f(x,w)$ where $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^{n_w}$, a zonotope $X = p \oplus H\mathbf{B}^m$ and a box W. Compute the following natural interval extensions:

- A zonotope $q \oplus S\mathbf{B}^n = \Box f(p,W)$.
- An interval matrix $M = \Box\nabla_x f(X,W)H$.
- A zonotope $\Psi(X,W) = q \oplus S\mathbf{B}^n \oplus \diamond M B^m = q \oplus H_q\mathbf{B}^l$ with $l = 2n+m$

Under previous assumptions, it results that $f(X,W) \subseteq \Psi(X,W)$

PROOF:

Given $w \in W$, the application of the mean value extension yields:

$$f(X,w) \subseteq f(p,w) \oplus (\nabla_x f(X,w))H\mathbf{B}^m$$

Thus,

$$f(X,W) \subseteq f(p,W) \oplus (\nabla_x f(X,W))H\mathbf{B}^m \subseteq$$
$$q + S\mathbf{B}^n \oplus \Box\nabla_x f(X,W)H\mathbf{B}^m =$$
$$q \oplus S\mathbf{B}^n \oplus M\mathbf{B}^m \subseteq$$
$$q \oplus S\mathbf{B}^n \oplus \diamond M\mathbf{B}^m = q \oplus H_q\mathbf{B}^l$$

■

Therefore, the operator of theorem 5 can be used like an inclusion function in the Step 1 of Algorithm 1. Note that this inclusion is not very conservative because it makes a sort of linearization of the range of the function. With this proposal, at each sample time, the order of the zonotope is increased. The computational cost increases quadratically, so it is interesting to dispose of an algorithm to bound a high order zonotope by a lower order one. This algorithm can be found in [8].

V. Bound on the consistent state set

In this section, a bound on the consistent state set is provided. This bound is obtained as the intersection of p strips in the state space. Given a measure $y_k \in \mathbb{R}^p$, the consistent state set was defined in section III as:

$$\mathcal{X}_{y_k} = \{\, x \in \mathbb{R}^n \,:\, y_k \in g(x,V) \,\}$$

Define now sets $\mathcal{X}_{y_k}(i)$, $i = 1,\ldots,p$, as the region of the state space consistent with the i-th component of output y_k:

$$\mathcal{X}_{y_k}(i) = \{\, x \in \mathbb{R}^n \,:\, y_k(i) \in g_i(x,V) \,\}$$

where $g_i(x,v)$ denotes the $i-th$ component of $g(x,v) \in \mathbb{R}^p$. With this definition it is clear that:

$$\mathcal{X}_{y_k} \subseteq \bigcap_{i=1}^{p} \mathcal{X}_{y_k}(i)$$

In what follows, we will show how to bound $\mathcal{X}_{y_k}(i)$ by means of a strip in the state space. Let us suppose that x_k is guaranteed to belong to zonotope $\bar{\mathcal{X}}_k$. Then, the $i-th$ component of the measured output y_k can be used to obtain a sharper bound of the state. In effect, $x_k \in \bar{\mathcal{X}}_k \cap \mathcal{X}_{y_k}(i)$. The following property shows that it is possible to bound $\bar{\mathcal{X}}_k \cap \mathcal{X}_{y_k}(i)$ by means of the intersection of $\bar{\mathcal{X}}_k$ and a strip in the state space.

Property 1: Let us suppose that zonotope $\bar{\mathcal{X}}_k$ and measured output y_k are given. Obtain, by means of interval arithmetic, vector $c_i \in \mathbb{R}^n$ and scalars $d_i, \sigma_i \in \mathbb{R}$ such that:

- $c_i = mid\left(\Box\nabla_x g_i(\bar{\mathcal{X}}_k, V)\right)$
- $c_i^\top \bar{\mathcal{X}}_k - g_i(\bar{\mathcal{X}}_k, \mathcal{V}) \subseteq [s_i - \sigma_i, s_i + \sigma_i]$

Then, defining $\mathcal{X}^e_{y_k}(i) = \{\, x \,:\, |c_i^\top x - y_k(i) - s_i| \leq \sigma_i \,\}$, it results that:

$$\bar{\mathcal{X}}_k \bigcap \mathcal{X}_{y_k}(i) \subseteq \bar{\mathcal{X}}_k \bigcap \mathcal{X}^e_{y_k}(i)$$

PROOF: In effect, if $x \in \bar{\mathcal{X}}_k \bigcap \mathcal{X}_{y_k}(i)$ then there exists $v \in V$ such that $y_k(i) = g_i(x,v)$. Multiplying the last inequality by -1 and adding $c_i^\top x$:

$$c_i^\top x - y_k(i) = c_i^\top x - g_i(x,v) \subseteq$$

$$c_i^\top \bar{X}_k - g_i(\bar{X}_k, V) \subseteq [s_i - \sigma_i, s_i + \sigma_i]$$

Therefore, $|c_i^\top x - y_k(i) - s_i| \le \sigma_i$. ■

VI. Guaranteed state intersection

Let us suppose that $x_{k-1} \in \tilde{X}_{k-1}$. Then, as it was stated in section 3, it is possible to bound the uncertain trajectory of the nonlinear system using interval arithmetic. In particular, a slight modification of kühn's method was proposed in order to obtain a zonotope $\bar{X}_k = \Psi(\tilde{X}_{k-1}, W)$ such that $f(\tilde{X}_{k-1}, W) \subseteq \bar{X}_k$. As it was demonstrated in the last section, the $i-th$ component of the measured output y_k can be used to obtain a strip $X^e_{y_k}(i)$ such that $x_k \in \bar{X}_k \cap X^e_{y_k}(i)$. Due to the fact that $\bar{X}_k$ is a zonotope and $X^e_{y_k}(i)$ a strip in the state space, it is convenient to obtain a procedure that bounds the intersection between a zonotope and a strip.

The next property provides a family of zonotopes (parameterized by means of vector λ) that contains the intersection of a zonotope and a strip. At the end of this section we will show how to choose parameter vector λ in order to minimize the volume of the obtained bound. Note that the results of this section can be applied for every component of the measured output y_k.

Property 2: Given the zonotope $X = p \oplus H\mathbf{B}^r \subset \mathbb{R}^n$, the strip $S = \{ x \in \mathbb{R}^n : |c^\top x - d| \le \sigma \}$ and the vector λ, define:

- $\hat{p}(\lambda) = p + \lambda(d - c^\top p)$
- $\hat{H}(\lambda) = \left[(I - \lambda c^\top)H \quad \sigma\lambda\right]$

Then,

$$X \bigcap S \subseteq \hat{X}(\lambda) = \hat{p}(\lambda) + \hat{H}(\lambda)\mathbf{B}^{r+1}$$

PROOF: Let us suppose that $x \in X \bigcap S$. Then $x \in X = p \oplus H\mathbf{B}^r$. This implies that there is $z \in \mathbf{B}^r$ such that:

$$x = p + Hz \tag{4}$$

In fact, adding and substracting $\lambda c^\top Hz$ to previous equality:

$$x = p + \lambda c^\top Hz + (\mathrm{I} - \lambda c^\top)Hz \tag{5}$$

From $x \in X \bigcap S$ it is inferred that $x \in S = \{ x \in \mathbb{R}^n : |c^\top x - d| \le \sigma \}$. Thus, there exists $w \in [-1, 1] = \mathbf{B}^1$ such that $c^\top x - d = \sigma w$. Taking into account equation (4) it results that $c^\top(p + Hz) - d = \sigma w$. That is:

$$c^\top Hz = d - c^\top p + \sigma w$$

Substituting this equality in equation (5), the following is obtained:

$$x = p + \lambda(d - c^\top p + \sigma w) + (I - \lambda c^\top)Hz =$$

$$p + \lambda(d - c^\top p) + \lambda\sigma w + (I - \lambda c^\top)Hz =$$

$$\hat{p}(\lambda) + \left[\ (I - \lambda c^\top)H \quad \sigma\lambda \ \right] \left[\begin{array}{c} z \\ w \end{array} \right] =$$

$$= \hat{p}(\lambda) + \hat{H}(\lambda) \left[\begin{array}{c} z \\ w \end{array} \right] \in \hat{X}(\lambda)$$

■

A. Minimizing the volume of the intersection

Let us suppose that we want to minimize the volume of $\hat{X}(\lambda)$. In this case, we should choose λ in such a way that the volume of the zonotope $\hat{X}(\lambda) = \hat{p}(\lambda) \oplus \hat{H}(\lambda)\mathbf{B}^{r+1}$ is minimized. That is, we are interested in the minimization of $\mathrm{Vol}\,(\hat{p}(\lambda) \oplus \hat{H}(\lambda)\mathbf{B}^{r+1})$. It is well known (see [18], [12]) that the volume of a zonotope $a \oplus D\mathbf{B}^m \subset \mathbb{R}^n$ is given by:

$$\mathrm{Vol}\,(a \oplus D\mathbf{B}^m) =$$
$$\sum_{i=1}^{N(n,m)} 2^n \left| \det \left[\ D_{s_1(i)} \quad D_{s_2(i)} \quad \ldots D_{s_n(i)} \ \right] \right|$$

where $N(n,m)$ denotes the number of different ways of choosing n elements from a set of m. D_i denotes the i-th column of D. Integers $s_j(i)$, $j = 1, \ldots, n$, $i = 1, \ldots, N$ denote each one of the different ways of choosing n elements from a set of m. That is, these integers satisfy:

$$1 \le s_1(i) < s_2(i) < \ldots < s_n(i) \le m$$

Moreover, if $i \ne j$ then:

$$\left[\ s_1(i) \quad \ldots \quad s_n(i) \ \right] \ne \left[\ s_1(j) \quad \ldots \quad s_n(j) \ \right]$$

Taking into account that $\hat{H}(\lambda) = \left[\ (\mathrm{I} - \lambda c^\top)H \quad \sigma\lambda \ \right]$, the expression corresponding to the volume of $\hat{X}(\lambda)$ is:

$$\mathrm{Vol}\,(\hat{p}(\lambda) \oplus \hat{H}(\lambda)\mathbf{B}^{r+1}) =$$
$$\sum_{i=1}^{N(n,m-1)} 2^n \left| \det \left[(\mathrm{I} - \lambda c^\top)A_i\right] \right|$$
$$+ \sum_{i=1}^{N(n-1,m-1)} 2^n \left| \det \left[\ (\mathrm{I} - \lambda c^\top)B_i \quad \sigma\lambda \ \right] \right|$$

Where A_i denotes each of the different matrices that can be obtained choosing n columns from matrix H. On the other hand, B_i denotes each of the different matrices that can be obtained choosing $n-1$ columns from H. Let us recall the following well known properties of the determinant of a matrix:

- $\det\,(AB) = \det\,(A) \det\,(B)$.
- Given vectors $a, b \in \mathbb{R}^n$: $\det\,(\mathrm{I} + ab^\top) = 1 + b^\top a$.

The above equalities will be used to calculate the different terms that appear in the expresion of the volume of $\hat{X}(\lambda)$. We will distinguish between two different classes of terms:

- Terms of the form $\det \left[\ (\mathrm{I} - \lambda c^\top)A_i \ \right]$:
  In this case,

  $$\det \left[\ (\mathrm{I} - \lambda c^\top)A_i \ \right] =$$
  $$\det\,(\mathrm{I} - \lambda c^\top) \det\,(A_i) = (1 - c^\top\lambda) \det\,(A_i)$$

- Terms of the form $\det \left[\ (\mathrm{I} - \lambda c^\top)B_i \quad \sigma\lambda \ \right]$:
  Note that $B_i - \lambda c^\top B_i$ is obtained substracting from each column of B_i the last column of $\left[(\mathrm{I} - \lambda c^\top)B_i \ \sigma\lambda\right]$ multiplied by a scalar. It is well known that the determinant of a matrix does not change if a column is added or substracted from another one. This implies that:

$$\det\begin{bmatrix}(\mathrm{I}-\lambda c^\top)B_i & \sigma\lambda\end{bmatrix} = \det\begin{bmatrix}B_i & \sigma\lambda\end{bmatrix}$$

Two different cases must be distinguished: $\text{rank}\,\{B_i\} < n-1$ and $\text{rank}\,\{B_i\} = n-1$. If $\text{rank}\,\{B_i\} < n-1$ then $\det\begin{bmatrix}B_i & \sigma\lambda\end{bmatrix} = 0$. In the following, it will be supposed that $\text{rank}\,\{B_i\} = n-1$. It is clear, under this assumption, that there exists v_i such that $v_i^\top v_i = 1$ and $v_i^\top * B_i = 0$. That is, v_i is orthonormal to $\text{Imag}\,(B_i)$. Therefore, $\Phi_i = \begin{bmatrix}B_i & v_i\end{bmatrix}$ is not singular. Note that:

$$\det\begin{bmatrix}B_i & \sigma\lambda\end{bmatrix} = \det\begin{bmatrix}B_i & (v_i - v_i + \sigma\lambda)\end{bmatrix} =$$

$$\det\left(\Phi_i + (\sigma\lambda - v_i)\begin{bmatrix}0 & 0 & \dots & 1\end{bmatrix}\right) =$$

$$\det(\Phi_i)\,\det\left(\mathrm{I} + \Phi_i^{-1}(\sigma\lambda - v_i)\begin{bmatrix}0 & 0 & \dots & 1\end{bmatrix}\right) =$$

$$\det(\Phi_i)(1 + \begin{bmatrix}0 & 0 & \dots & 1\end{bmatrix}\Phi_i^{-1}(\sigma\lambda - v_i))$$

Taking into account that $\begin{bmatrix}0 & 0 & \dots & 1\end{bmatrix}\Phi_i^{-1} = v_i^\top$:

$$\det(\Phi_i)(1 + v_i^\top(\sigma\lambda - v_i)) = \det(\Phi_i)(1 + \sigma v_i^\top\lambda - 1) =$$

$$\sigma\,\det(\Phi_i)(v_i^\top\lambda)$$

To conclude, the volume of $\hat{\mathcal{X}}(\lambda)$ is given by the following expression:

$$\text{Vol}\,(\hat{p}(\lambda) \oplus \hat{H}(\lambda)\mathbf{B}^{r+1}) =$$

$$\sum_{i=1}^{N(n,m-1)} 2^n |1 - c^\top\lambda|\,|\det(A_i)|$$

$$+ \sum_{i=1}^{N(n-1,m-1)} \sigma 2^n \left|\det\begin{bmatrix}B_i & v_i\end{bmatrix}\right| |v_i^\top\lambda|$$

Note that $\text{Vol}\,(\hat{p}(\lambda) \oplus \hat{H}(\lambda)\mathbf{B}^{r+1})$ is a convex function of λ. This means that obtaining the vector λ that minimizes the volume of the zonotope is a convex problem. Therefore, specialized algorithms can be used.

VII. Examples

A. Example 1

A benchmark problem is considered [4]. The system is described by:

$$x_{k+1} = \begin{bmatrix}0 & -0.5\\ 1 & 1+0.3\delta_k\end{bmatrix}x_k + 0.02\begin{bmatrix}-6\\ 1\end{bmatrix}w_k$$

$$y_k = \begin{bmatrix}-2 & 1\end{bmatrix}x_k + 0.2v_k$$

with $|\delta_k| \le 1, \|w_k\|_\infty \le 1, \|v_k\|_\infty \le 1$. The initial state belongs to the box $3\mathbf{B}^2$. The signal to be estimated is $z_k = \begin{bmatrix}1 & 0\end{bmatrix}x_k$. The order of zonotopes are limited to $m \le 20$.

Fig. 1 shows a succession of $\bar{\mathcal{X}}_k$ sets and how the proposed algorithm reduces their volumes by intersection, obtaining $\tilde{\mathcal{X}}_k$ sets. Fig. 2 demonstrates that the algorithm provides a guaranteed bound of the actual state of the system.

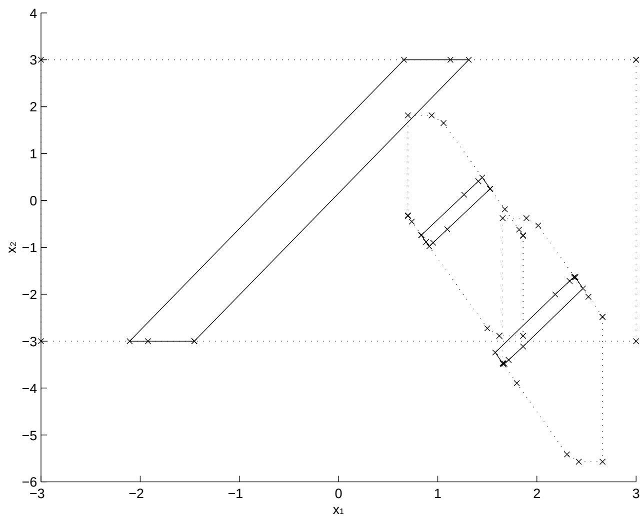

Fig. 1. Estimation by zonotopes. Dotted lines represent $\bar{\mathcal{X}}_k$ sets and solid lines represent $\tilde{\mathcal{X}}_k$ sets.

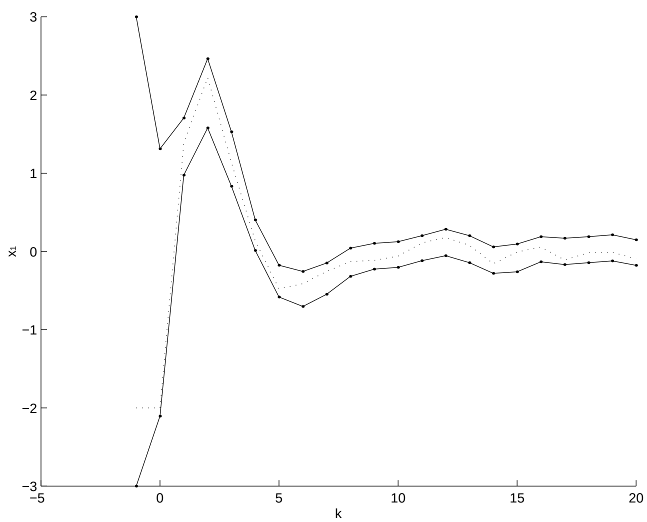

Fig. 2. The dotted line represents z_k and the solid lines represent the bounds of z_k obtained by the presented algorithm .

B. Example 2

A nonlinear estimation example is presented here. An isothermal gas-phase [19] reactor is charged with an initial amount of A and B, and the species are allowed to react according to the reversible reaction. The goal is to reconstruct the partial pressure of each specie in the reactor using the measurements of the total pressure of the vessel as the reaction proceeds. The system is modelled by:

$$\dot{x}_1 = -2k_1x_1^2 + 2k_2x_2$$

$$\dot{x}_2 = k_1x_1^2 - k_2x_2$$

in which $k_1 = 0.16\,min^{-1}atm^{-1}, k_2 = 0.0064\,min^{-1}$. The measured output is the total pressure described by: $y_k = \begin{bmatrix}1 & 1\end{bmatrix}x_k + v_k$ where $\|v_k\|_\infty \le 0.3$. The initial conditions are

$$x_0 \in \begin{bmatrix}2.5\\ 1.0\end{bmatrix} \oplus \begin{bmatrix}2.5 & 0\\ 0 & 0.5\end{bmatrix}\mathbf{B}^2$$

The sampling time is 6 seconds. Note that an Extended Kalman Filter fails for this example [19] because more than one equilibrium point may satisfy the output equation. All states are estimated. The order of zonotopes are limited to $m \leq 20$. Fig. 3 shows the evolution of the volume of the guaranteed bound of the state.

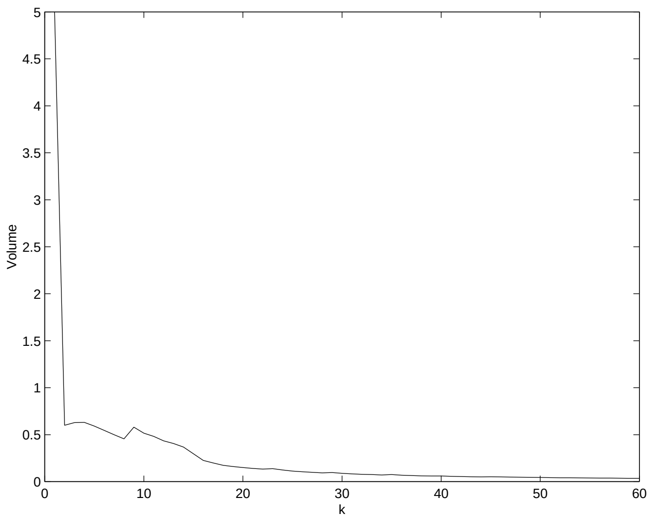

Fig. 3. Evolution of the volume of the guaranteed bound of the state

VIII. Conclusions

A new approach to guaranteed state estimation for non-linear discrete-time systems with a bounded description of noise and parameters has been proposed. The algorithm computes a set of all states consistent with the measured output and the given noise and parameters. This set is represented by a zonotope and is calculated by interval arithmetic. Its volume is minimized each sample instant resolving a convex optimization problem. Two examples have been provided for clarifying the algorithm.

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